# UNCLASSIFIED

## AD 445028

# DEFENSE DOCUMENTATION CENTER

FOR

## SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

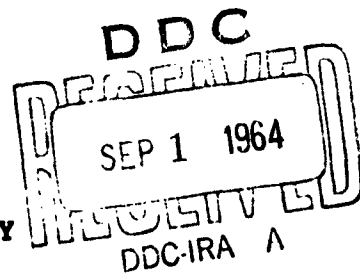# UNCLASSIFIED

Technical Note N-598

DETERMINATION OF DAMPING CONSTANTS FOR A DRY

FRICTION-VISCOUS DAMPED OSCILLATOR

BY

D. A. DaDeppo

3 August 1964

U. S. NAVAL CIVIL ENGINEERING LABORATORY
Port Hueneme, California

DETERMINATION OF DAMPING CONSTANTS FOR A
DRY FRICTION-VISCOUS DAMPED OSCILLATOR *

Y-F008-08-02-102

Type C

by

D. A. DaDeppo
Associate Professor of Engineering Mechanics, University of Detroit

ABSTRACT

The method of least squares is applied to the problem of
analyzing a decay record to determine the damping constants for
a dry friction-viscous damped, single-degree-of-freedom system.
Solution of the set of non-linear equations which yield the constants
is obtained by applying the Newton-Raphson method of iteration.
Sample calculations show that the method is not well-suited for
manual computation.

A program is presented for calculating the damping con-
stants by means of a digital computer.

## CONTENTS

## INTRODUCTION

The objective of this work was to develop an analytical method for analyzing an experimentally obtained decay record to determine the damping constants for a dry friction-viscous damped single-degree-of-freedom system.

A semi-graphical, trial and error method of analysis is available;[1] however, as pointed out by Jacobsen and Ayre,[1] a precise determination of the damping constants by that method is time consuming. Moreover, engineering judgment must be used to determine when the constants have been calculated with sufficient accuracy; and, the semi-graphical procedure may lead to inconsistent results.

In this report the method of least squares is applied to the problem of analyzing a vibration trace. The method yields consistent results, but is not well-suited for manual computation. A program is presented for carrying out the calculations by means of a digital computer.

## ANALYTICAL DEVELOPMENT

The present study is limited to linear systems in which the damping forces are due to a combination of viscous friction and dry friction. The objective of the study is to determine the damping constants from an experimentally obtained decay curve. As indicated in Figure 1, the experimental curve for such a system will not, in general, be an exact representation of the theoretically correct curve; and the direction, but not the precise location, of the time axis of the experimental curve is known.

The generally accepted procedure employed in analyzing a decay record is to first construct a theoretical curve that is a "good" approximation to the experimental curve. Then, it is assumed that the damping constants for the system are the damping constants which appear in the set of parameters that define the theoretical curve. Clearly, it is essential that a definition of what constitutes a good approximation be given in terms of measurable quantities. For simplicity the goodness of an approximation is measured in terms of peak displacements on the theoretical and experimental decay curves.
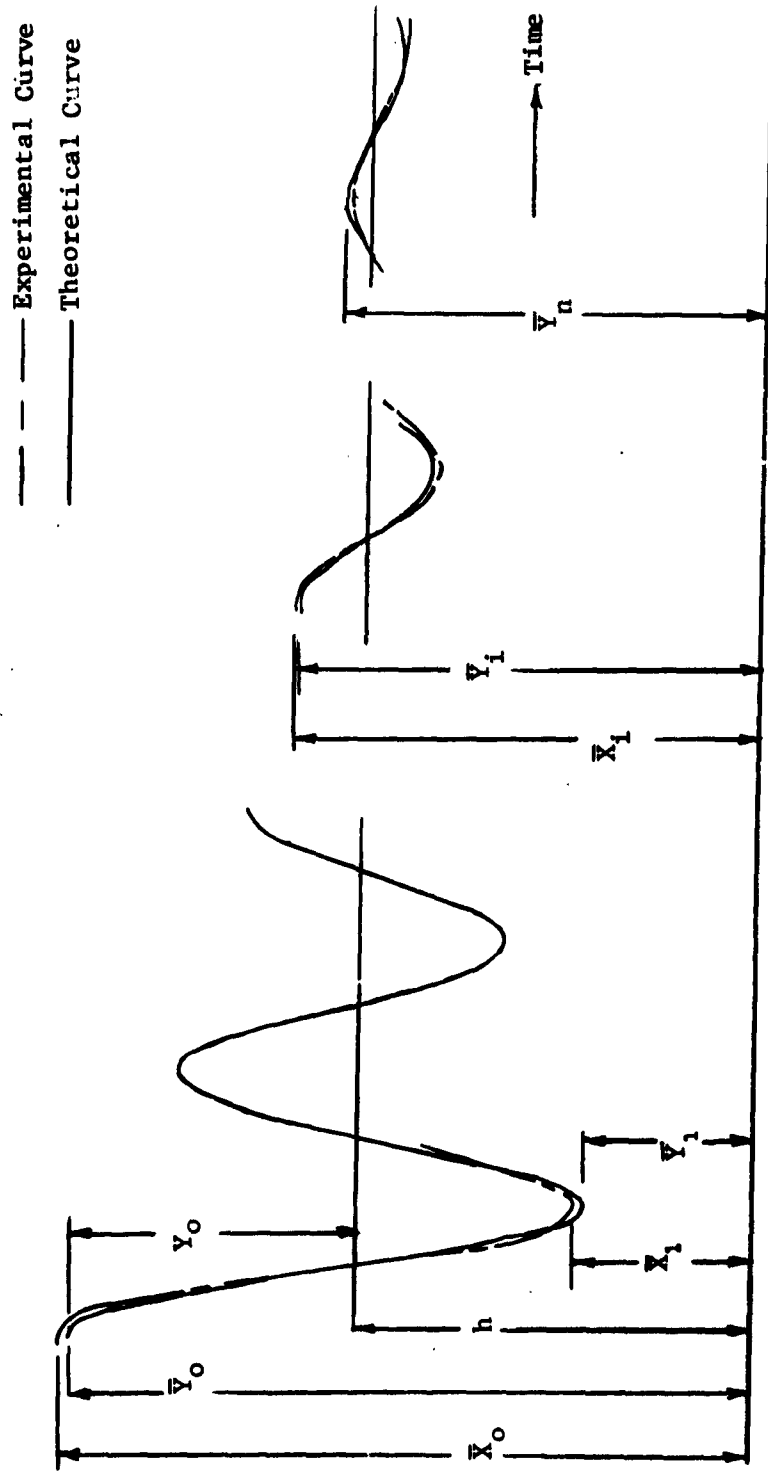
Figure 1. Decay curves.

2

## Statement of the Problem

With the preceding discussion as background the problem may be stated as follows.

With reference to Figure 1, let $\bar{X}_0$, $\bar{X}_1$, ..., $\bar{X}_n$ be peak displacements on an experimental decay curve, where $\bar{X}_i$ is measured from an axis which is parallel to, but not necessarily coincident with the true steady state displacement axis; let $\bar{Y}_0$, $\bar{Y}_1$, ..., $\bar{Y}_n$ be the corresponding peaks on a theoretical curve defined by the equations

$$\bar{Y} = Y + h \tag{1}$$

$$\ddot{Y} + 2\nu p \dot{Y} + p^2[Y + (\text{sgn } \dot{Y})\Delta] = 0 \tag{2}$$

$$Y(0) = Y_0, \quad \dot{Y}(0) = 0 \tag{3}$$

and, determine the distance h, the initial displacement $Y_0$, and the damping constants $\nu$ and $\Delta$ such that the "square error"

$$E = E(\nu, \Delta, Y_0, h) = \frac{1}{2} \sum_{i=0}^{n} (\bar{X}_i - \bar{Y}_i)^2 \tag{4}$$

is a minimum.

To solve the problem, one must first determine how $\bar{Y}_i$ depends on the parameters appearing in Equations 1, 2, and 3. This is done in the following section.

## Decay Curve Analysis

The extreme values of the displacement Y(t) which are obtained by solving Equation 2, occur at equally spaced intervals of time, and it is not difficult to show that the peak displacements are given by the formulas

$$Y_i = Y_0 \quad (i = 0) \tag{5a}$$

$$Y_i = \Delta (1 + \delta)(-1)^{i-1} - \delta Y_{i-1} \quad (i \neq 0) \tag{5b}$$

where

$$\delta = e^{\frac{-v\pi}{\sqrt{1-v^2}}} \tag{6}$$

and

$v$ = damping ratio for viscous damping

$\Delta$ = damping constant for dry friction damping

For future use, it is noted here that Equation 5 may be rewritten in the form

$$Y_i = \Delta(1 + \delta)a_i + Y_o b_i \tag{5c}$$

where

$$a_o = 0 \quad ; \quad a_i = (-1)^{i-1} \sum_{k=0}^{i-1} \delta^k$$

$$b_o = 1 \quad ; \quad b_i = (-1)^i \delta^i$$

Equations 1 and 5 may be combined to obtain

$$\overline{Y}_o = Y_o + h \tag{7a}$$

$$\overline{Y}_i = \Delta(1 + \delta)(-1)^{i-1} - \delta(\overline{Y}_{i-1} - h) + h \tag{7b}$$

In view of Equation 6, it follows that the relationships expressed by Equation 7 show how $\overline{Y}_i$ depends on the damping constants, the initial displacement $Y_o$, and h. It should be noted that $\overline{Y}_i$ is independent of p (see Equation 2).

The solution of Equation 7 is of interest. Setting i = k, i = k + 2j, i = k + 2j + 1, where j is an integer, in Equation 7b and combining the resulting equations gives

$$\delta = - (\overline{Y}_k - \overline{Y}_{k+2j})(\overline{Y}_{k-1} - \overline{Y}_{k+2j-1})^{-1} \tag{8}$$

4

$$\Delta = \frac{(-1)^{k-1}}{1+\delta} \left[ (Y_k - Y_{k+2j+1}) + \delta(Y_{k-1} - Y_{k+2j}) \right] \qquad (9)$$

$$h = -\Delta(-1)^{i-1} + (Y_k + \delta Y_{k-1})(1 + \delta)^{-1} \qquad (10)$$

and, from Equation 7a,

$$Y_o = Y_o - h \qquad (11)$$

Equations 8 and 9 can be used to estimate the damping constants for the system characterized by the experimental decay curve by substituting $X_k$ for $Y_k$.

Equations 8 to 11 show that $\delta, \Delta$, $Y_o$ and h are invariant functions of the peak displacements on the theoretical decay curve. Therefore, if the invariance is not preserved when $Y_k$ is replaced by $X_k$ then it is known that the experimental decay curve does not coincide with the theoretical curve.

Calculation of Damping Constants by the Method of Least Squares

Nothing is lost and the subsequent derivations are simplified if $\nu$ is replaced by $\delta$ in Equation 4. Then, the equation becomes

$$E(\delta, \Delta, Y_o, h) = \frac{1}{2} \sum (X_i - Y_i)^2 \qquad (12)$$

The variables $\delta$, $\Delta$, $Y_o$, h must satisfy necessary conditions for E to be a minimum. These conditions are,

$$E,_\delta = F_\delta(\delta, \Delta, Y_o, h) = 0 \qquad (13a)$$

$$E,_\Delta = F_\Delta(\delta, \Delta, Y_o, h) = 0 \qquad (13b)$$

$$E,_{Y_o} = F_{Y_o}(\delta, \Delta, Y_o, h) = 0 \qquad (13c)$$

$$E,_h = F_h(\delta, \Delta, Y_o, h) = 0 \qquad (13d)$$

where the comma notation has been used to indicate partial derivatives (i.e., $E,_\delta = \partial E/\partial \delta$, etc.).

Equations 13 are linear in $\Delta$, $Y_0$ and h, but nonlinear in $\delta$; and they are not amenable to solution by analytical methods. However, the equations can be solved numerically by the Newton-Raphson Method for simultaneous equations.[2]

To derive the pertinent equations, and to outline the procedure, let $\delta^o$, $\Delta^o$, $Y_0^o$, $h^o$ be approximate roots; and, let $d\delta$, $d\Delta$, $dY_0$, $dh$ be corrections so that

$$\delta = \delta^o + d\delta \tag{14a}$$

$$\Delta = \Delta^o + d\Delta \tag{14b}$$

$$Y_0 = Y_0^o + dY_0 \tag{14c}$$

$$h = h^o + dh \tag{14d}$$

Then, Equations 13 become

$$F_\delta \ (\delta^o + d\delta, \ . \ . \ . \ , \ h^o + dh) = 0 \tag{15a}$$

$$F_\Delta \ (\delta^o + d\delta, \ . \ . \ . \ , \ h^o + dh) = 0 \tag{15b}$$

$$F_{Y_0} \ (\delta^o + d\delta, \ . \ . \ . \ , \ h^o + dh) = 0 \tag{15c}$$

$$F_h \ (\delta^o + d\delta, \ . \ . \ . \ , \ h^o + dh) = 0 \tag{15d}$$

Expanding each of Equations 15 by Taylor's theorem for a function of four variables, and discarding all terms containing products and/or powers of $d\delta$, $d\Delta$, etc., leads to the following equations to be solved for the "first corrections," $d\delta^1$, $d\Delta^1$, etc.

$$F_\delta^o + (E_{,\delta\delta})^o d\delta^1 + (E_{,\Delta\delta})^o d\Delta^1 + (E_{,Y_0\delta})^o dY_0^1 + (E_{,h\delta})^o dh^1 = 0 \tag{16a}$$

$$F_\Delta^o + (E_{,\delta\Delta})^o d\delta^1 + (E_{,\Delta\Delta})^o d\Delta^1 + (E_{,Y_0\Delta})^o dY_0^1 + (E_{,h\Delta})^o dh^1 = 0 \tag{16b}$$

6

$$F_{Y_o}^o + (E_{,\delta Y_o})^o d\delta^1 + (E_{,\Delta Y_o})^o d\Delta^1 + (E_{,Y_o Y_o})^o dY_o^1 + (E_{,h Y_o})^o dh^1 = 0 \quad (16c)$$

$$F_h^o + (E_{,\delta h})^o d\delta^1 + (E_{,\Delta h})^o d\Delta^1 + (E_{,Y_o h})^o dY_o^1 + (E_{,hh})^o dh^1 = 0 \quad (16d)$$

where the superscript zero indicates that the quantity is to be evaluated at the "point" $(\delta^o, \Delta^o, Y_o^o, h^o)$

Now, the improved roots are

$$\delta^1 = \delta^o + d\delta^1 \quad (17a)$$

$$\Delta^1 = \Delta^o + d\Delta^1 \quad (17b)$$

$$Y_o^1 = Y_o^o + dY_o^1 \quad (17c)$$

$$h^1 = h^o + dh^1 \quad (17d)$$

Additional corrections are found by repeated applications of Equations 13 and 16.

All possible combinations of first and second partial derivatives of $Y_i$ are needed to evaluate the differential coefficients appearing in Equations 16. To calculate the derivatives we first combine Equations 1 and 5c to obtain

$$Y_i = \Delta(1 + \delta)a_i + Y_o b_i + h \quad (18)$$

Then, since $a_i$ and $b_i$ are functions of $\delta$, differentiation of Equation 18 gives (using primes to indicate differentiation with respect to $\delta$)

$$Y_{i,\delta} = \Delta[(1 + \delta)a_i' + a_i] + Y_o b_i' \quad (19)$$

$$Y_{i,\Delta} = (1 + \delta)a_i \quad (20)$$

$$Y_{i,Y_o} = b_i \quad (21)$$

$$Y_{i,h} = 1 \quad (22)$$

7

$$Y_{i,\delta\delta} = \Delta\left[(1+\delta)a_i'' + 2a_i'\right] + Y_0 b_i'' \tag{23}$$

$$Y_{i,\Delta\delta} = (1+\delta)a_i' + a_i \tag{24}$$

$$Y_{i,Y_0\delta} = b_i' \tag{25}$$

All other second partial derivatives of $Y_i$ are zero.

Evaluation of the derivatives is simplified by using recurrence formulas which follow from the definitions of $a_i$ and $b_i$ (see Equation 5c). It is easy to establish the following relations

$$a_0 = 0 \quad ; \quad a_i = -a_{i-1} + b_{i-1} \tag{26}$$

$$b_0 = 1 \quad ; \quad b_i = -\delta b_{i-1} \tag{27}$$

$$a_0' = 0 \quad ; \quad a_i' = -a_{i-1}' + b_{i-1}' \tag{28}$$

$$b_0' = 0 \quad ; \quad b_i' = -b_{i-1} - \delta b_{i-1}' \tag{29}$$

$$a_0'' = 0 \quad ; \quad a_i'' = -a_{i-1}'' + b_{i-1}'' \tag{30}$$

$$b_0'' = 0 \quad ; \quad b_i'' = -2b_{i-1}' - \delta b_{i-1}'' \tag{31}$$

The first partial derivatives of $E$ are obtained from Equation 12. The derivatives are

$$E_{,\delta} = -\Sigma z_i Y_{i,\delta} = F_\delta \tag{32}$$

$$E_{,\Delta} = -\Sigma z_i Y_{i,\Delta} = F_\Delta \tag{33}$$

$$E_{,Y_0} = -\Sigma z_i Y_{i,Y_0} = F_{Y_0} \tag{34}$$

$$E_{,h} = -\Sigma z_i Y_{i,h} = F_h \tag{35}$$

8

where all summations are from $i = 0$ to $i = n$ and $Z_i = X_i - \bar{Y}_i$. The second partial derivatives appearing in Equations 16 are

$$E_{,\delta\delta} = \Sigma \left[ - Z_i \bar{Y}_{i,\delta\delta} + (\bar{Y}_{i,\delta})^2 \right] \tag{36}$$

$$E_{,\Delta\delta} = \Sigma \left[ - Z_i \bar{Y}_{i,\Delta\delta} + \bar{Y}_{i,\Delta}\bar{Y}_{i,\delta} \right] = E_{,\delta\Delta} \tag{37}$$

$$E_{,Y_0\delta} = \Sigma \left[ - Z_i \bar{Y}_{i,Y_0\delta} + \bar{Y}_{i,Y_0}\bar{Y}_{i,\delta} \right] = E_{,\delta Y_0} \tag{38}$$

$$E_{,h\delta} = \Sigma \, \bar{Y}_{i,h} = E_{,\delta h} \tag{39}$$

$$E_{,\Delta\Delta} = \Sigma \, (\bar{Y}_{i,\Delta})^2 \tag{40}$$

$$E_{,Y_0\Delta} = \Sigma \, \bar{Y}_{i,Y_0}\bar{Y}_{i,\Delta} = E_{,\Delta Y_0} \tag{41}$$

$$E_{,h\Delta} = \Sigma \, \bar{Y}_{i,\Delta} = E_{,\Delta h} \tag{42}$$

$$E_{,Y_0 Y_0} = \Sigma \, (\bar{Y}_{i,Y_0})^2 \tag{43}$$

$$E_{,hY_0} = \Sigma \, \bar{Y}_{i,Y_0} = E_{,Y_0 h} \tag{44}$$

$$E_{,hh} = n + 1 \tag{45}$$

where all summations are from $i = 0$ to $i = n$, and use has been made of the fact that certain second partial derivatives of $\bar{Y}_i$ are identically zero.

Because of the special nature of Equations 33, 34, and 35, it is always possible to determine $\Delta$, $Y_0$ and $h$ as functions of $\delta$ such that $F_\Delta$, $F_{Y_0}$, and $F_h$ are identically zero, as required by the necessary conditions (Equations 13) for $E$ to be a minimum. This fact can be used to advantage in calculating the roots of Equations 13. On setting $F_\Delta$, $F_{Y_0}$, and $F_h$ equal to zero, combining Equations 20, 21, and 22 with Equations 33, 34, and 35 and rearranging terms, one obtains the equations

$$\Delta(1 + \delta) \, \Sigma \, a_i^2 + Y_0 \, \Sigma \, a_i b_i + h \, \Sigma \, a_i = \Sigma \, X_i a_i \tag{46}$$

9

$$\Delta(1 + \delta) \Sigma a_i b_i + Y_0 \Sigma b_i^2 + h \Sigma b_i = \Sigma X_i b_i \qquad (47)$$

$$\Delta(1 + \delta) \Sigma a_i + Y_0 \Sigma b_i + h(n + 1) = \Sigma X_i \qquad (48)$$

where all summations extend from $i = 0$ to $i = n$. It should be observed that Equations 46 to 48 are nothing more than Equations 13b, 13c, and 13d, but written in an alternate form.

## Numerical Procedure

Equations 8 to 11 can be used to calculate the approximate roots needed to start the numerical solution. However, trial calculations show that it is better to calculate $\delta$ from Equation 8 (using, of course, $X_k$ in place of $Y_k$), and then calculate $\Delta^\circ$, $Y_0^\circ$, and $h^\circ$ by writing and solving Equations 46 to 48. The procedure used to solve the problem is as follows:

1. Use Equation 8 to calculate $\delta^\circ$.

2. Calculate $\Delta^\circ$, $Y_0^\circ$, and $h^\circ$ by writing and solving Equations 46 to 48.

3. Calculate $F_\delta^\circ$ using Equation 32.

4. If $F_\delta^\circ = 0$, the solution has been found, since $F_\Delta^\circ$, $F_{Y_0}^\circ$, and $F_h^\circ$ are zero by construction.

5. If $F_\delta^\circ \neq 0$ calculate the differential coefficients in Equations 16 by applying Equations 19 to 31, and Equations 36 to 45.

6. Solve Equations 16 for $d\delta^1$.

7. Replace $\delta^\circ$ by $\delta^\circ + d\delta^1$ and repeat the process.

## Damping Ratio and Logarithmic Decrement

The damping ratio and the logarithmic decrement are measures of the amount of viscous damping in a system. Let $\ell$ be the logarithmic decrement. It is shown in elementary vibrations that

$$\ell = \frac{2\pi v}{\sqrt{1-v^2}} \qquad (49)$$

where $v$ is the damping ratio. In view of Equation 6 we have

10

$$\ell = 2\text{Ln}(\delta^{-1}) \qquad\qquad (50)$$

and, from Equation 49

$$\nu = \frac{\ell}{\sqrt{\ell^2 + 4\pi^2}} \qquad\qquad (51)$$

Discussion

Sample calculations demonstrating the use of the theory are presented in Appendix A. In the example, convergence of the iterative procedure was rapid; two iterations were required. The decay record used in the example was taken from Reference 1. Because the calculations are lengthy, only the first six (of thirteen) peaks were used in the analysis.

The same decay curve was analyzed by means of a digital computer, using six peaks and using twelve peaks. Computer results are presented in Appendix B. A comparison of the results indicates that an analysis based on a partial or incomplete decay record may not lead to accurate values of the damping constants.

It is essential to note that an experimental decay curve may deviate from the theoretical curve for a linear system with viscous friction and dry friction damping because (a) the recorded decay curve is not an accurate representation of the theoretical curve; or (b) other forms of damping are present; or because of (a) and (b). The methods presented in this report are strictly applicable only when deviation can be assigned to cause (a). The methods may be applied when other forms of damping are present, if it is assumed that the calculated values of δ and Δ represent the contribution of viscous friction forces and dry friction forces to the total damping force.

CONCLUSION

It has been shown how the method of least squares can be applied to the problem of analyzing a decay record to calculate damping constants. Sample calculations show that the method is not well-suited for manual computation.

11

## REFERENCES

1. Jacobsen, L. S., and Ayre, R. S. "Engineering Vibrations,"
McGraw-Hill Book Co., Inc., 1958.

2. Scarborough, J. B., "Numerical Mathematical Analysis," Fourth
Ed., The Johns Hopkins Press, Baltimore, Maryland, 1958.

## Appendix A

## SAMPLE CALCULATIONS

In this appendix the theory is applied to the decay record shown in Figure A-1. The decay record is a reproduction of the curve shown in Figures 5-13, page 216, in the text "Engineering Vibrations," by L. S. Jacobsen and R. S. Ayre. Because the calculations are lengthy, only the first six peaks on the decay curve were considered. The procedure given on page 10 was used in solving the problem. The computations for the first cycle of the procedure are carried out below. Where possible, numerical results are presented in tabular form.

Calculation of $\delta^o$

Using Equation 8 with $k = 1$, $j = 1$ we obtain

$$\delta^o = -(-4.84 + 3.56)(5.60 - 4.17)^{-1} = 0.8951$$

and, with $k = 2$, $j = 1$ we have

$$\delta^o = -(4.17 - 3.01)(-4.84 + 3.56)^{-1} = 0.9062$$

The average value is approximately 0.9, and $\delta^o = 0.9$ is used to start the solution. The calculation of $a_i$, $b_i$, and their derivatives is displayed in Table A-1.

13
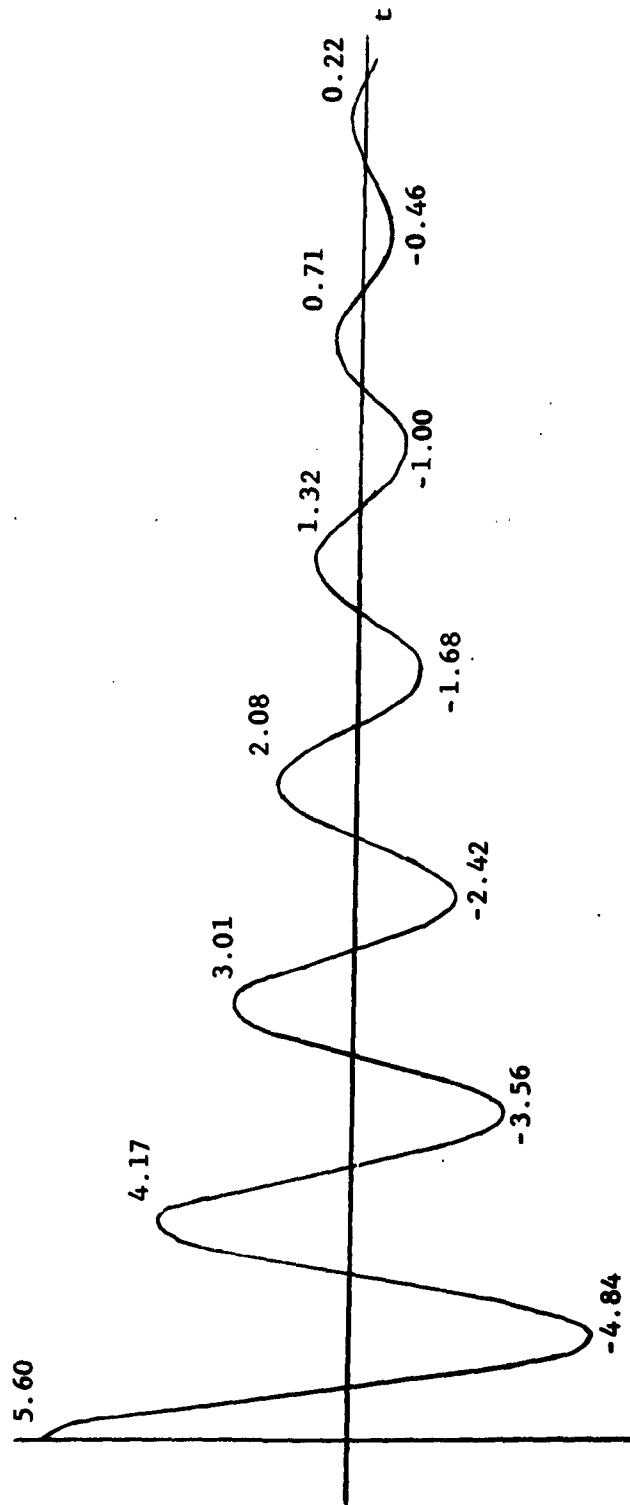
Figure A-1.  Decay curve.

(Reference:  Engineering Vibrations, Jacobsen and Ayre, 1958)

5.60  4.17  3.01  2.08  1.32  0.71  0.22  t

-4.84  -3.56  -2.42  -1.68  -1.00  -0.46

Table A-1. Computation of $a_i$, $b_i$, and Associated Derivatives

$$\delta^o = 0.9$$

| Column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
|---|---|---|---|---|---|---|---|---|
| Equation | - | - | 26 | 27 | 28 | 29 | 30 | 31 |
| Quantity | $i$ | $\overline{X}_i$ | $a_i$ | $b_i$ | $a_i'$ | $b_i'$ | $a_i''$ | $b_i''$ |
| | 0 | 5.60 | 0 | 1.0 | 0 | 0 | 0 | 0 |
| | 1 | -4.84 | 1.0 | -0.9 | 0 | -1.0 | 0 | 0 |
| | 2 | 4.17 | -1.9 | 0.81 | -1.0 | 1.8 | 0 | 2.0 |
| | 3 | -3.56 | 2.71 | -0.729 | 2.8 | -2.43 | 2.0 | -5.4 |
| | 4 | 3.01 | -3.439 | 0.6561 | -5.23 | 2.916 | -7.4 | 9.72 |
| | 5 | -2.42 | 4.0951 | -0.5905 | 8.146 | -3.2805 | 17.12 | -14.58 |

Equations 46 to 48 become

$$40.5507 \, \Delta^o(1 + \delta^o) - 9.0891 \, Y_o^o + 2.4661 \, h^o = -42.6721$$

$$-9.0891 \, \Delta^o(1 + \delta^o) + 3.7767 \, Y_o^o + 0.2466 \, h^o = 19.6721$$

$$2.4661 \, \Delta^o(1 + \delta^o) + 0.2466 \, Y_o^o + 6 \, h^o = 1.96000$$

and the solution is

$$\Delta^o(1 + \delta^o) = 0.2044$$

$$Y_o^o = 5.6101$$

$$h^o = 0.0121$$

$$\Delta^o = 0.1076$$

## Calculation of First Corrections

The basic quantities needed to calculate the first corrections are shown in Table A-2.

Substituting the quantities appearing in columns 5 and 10 into Equation 32 gives

$$F_\delta^o = -0.1139$$

and by the selection of $\Delta^o$, $Y_o^o$ and $h^o$,

$$F_\Delta^o = F_{Y_o}^o = F_h^o = 0$$

Now, Equations 16 become

$$776.6387 \; d\delta^1 - 335.7554 \; d\Delta^1 + 41.5069 \; dY_o^1 - 9.9599 \; dh^1 = 0.1139$$

$$-335.7554 \; d\delta^1 + 146.3881 \; d\Delta^1 - 17.2692 \; dY_o^1 + 4.6856 \; dh^1 = 0$$

$$41.5069 \; d\delta^1 - 17.2692 \; d\Delta^1 + 3.7767 \; dY_o^1 + 0.2466 \; dh^1 = 0$$

$$9.9599 \; d\delta^1 + 4.6856 \; d\Delta^1 + 0.2466 \; dY_o^1 + 6 \; dh^1 = 0$$

And, the solution for $d\delta^1$ is

$$d\delta^1 = 0.0254$$

So, the approximate value of $\delta$ to be used in the second cycle of iteration is

$$\delta^1 = \delta^o + d\delta^1 = 0.9254$$

Table A-2. $\Psi_i$ and Associated Partial Derivatives

$$\delta^o = 0.9$$

| Column (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---|---|---|---|---|---|---|---|---|---|
| Equation - | - | 18 | 19 | 20 | 21 | 23 | 24 | 25 | - |
| Quantity i | $X_i$ | $\Psi_i$ | $\Psi_{i,\delta}$ | $\Psi_{i,\Delta}$ | $\Psi_{i,Y_o}$ | $\Psi_{i,\delta\delta}$ | $\Psi_{i,\Delta\delta}$ | $\Psi_{i,Y_o\delta}$ | $\Psi_i - X_i$ |
| 0 | 5.60 | 5.6222 | 0. | 0. | 1. | 0. | 0. | 0. | 0.0222 |
| 1 | -4.84 | -4.8326 | -5.5025 | 1.9 | -0.9 | 0. | 1. | -1. | 0.0074 |
| 2 | 4.17 | 4.1679 | 9.6894 | -3.61 | 0.81 | 11.0050 | -3.8 | 1.8 | -0.0021 |
| 3 | -3.56 | -3.5237 | -12.7686 | 5.149 | -0.729 | -29.2833 | 8.03 | -2.43 | 0.0363 |
| 4 | 3.01 | 2.9899 | 14.9199 | -6.5341 | 0.6561 | 51.8922 | -13.376 | 2.916 | -0.0201 |
| 5 | -2.42 | -2.4636 | -16.2981 | 7.7807 | -0.5905 | -76.5428 | 19.5725 | -3.2805 | -0.0436 |

## Subsequent Calculations

In the second cycle, with the above value for $\delta^1$, it is found by writing and solving Equations 46 to 48 that

$$Y_0^1 = 5.5820575$$

$$h^1 = 0.0124610$$

$$\Delta^1 = 0.1618079$$

and

$$F_\delta^1 = -0.002236$$

The complete solution of Equations 16 in the second cycle of iteration is

$$d\delta^2 = 0.5346 \times 10^{-3}$$

$$d\Delta^2 = 1.1264 \times 10^{-3}$$

$$dY_0^2 = -0.5928 \times 10^{-3}$$

$$dh^2 = 0.0081 \times 10^{-3}$$

At the end of the second cycle of iteration, the new approximate value of $\delta$ is

$$\delta^2 = \delta^1 + d\delta^2 = 0.9259346$$

With the above value for $\delta$ the solution of Equations 46 to 48 is

$$\Delta^2 = 0.1629342$$

$$Y_0^2 = 5.5814647$$

$$h^2 = 0.0124691$$

and it is found that

$$F_\delta^2 = -0.36 \times 10^{-5}$$

18

$F_\delta^2$ is sufficiently small to justify termination of the calculation. Table A-3 gives a summary of the successive approximations to the damping constants.

Table A-3.  Summary of Successive Approximations

| Cycle | $\delta$ | per cent change | $\Delta$ | per cent change |
|---|---|---|---|---|
| 0 | 0.9 | --- | 0.1076 | --- |
| 1 | 0.9254 | +2.82 | 0.1618079 | +50.4 |
| 2 | 0.9259346 | +0.06 | 0.1629342 | + 0.7 |

The computer program presented in Appendix B was used to solve the example problem, and to analyze the decay curve shown in Figure A-1 using the first twelve peaks. Table A-4 shows the values for $\delta$ as calculated by hand, and as calculated with the IBM 1620 digital computer.

Table A-4.  Comparison of Calculated Values of $\delta$ and $v$

| No. of Peaks N | Manual Solution $\delta$ | Computer Solution | |
|---|---|---|---|
| | | $\delta$ | $v$ |
| 6 | 0.9259346 | 0.9259393 | 0.0244855 |
| 12 | --- | 0.8952082 | 0.0352147 |

Consider, now, the per cent change in $v$. From Table A-4, this change is calculated to be 30.5%.

## Appendix B

## COMPUTER PROGRAM

The computer program was written in the Fortran language and is shown on pages 21 to 24. The program was designed to analyze a decay record with a maximum of 30 peaks. In the computer the iteration procedure is terminated when $d\delta^{k+1} / \delta^k \leq 10^{-5}$ or when $k = 100$ (the maximum number of iterations).

Listings of typical sets of input data are shown on page 25 and the output is shown on page 26. Printing of the $Y_i$ can be suppressed at the option of the user by placing a zero punch in column 12 of the control card. If the iteration procedure fails to converge within 100 cycles, the output consists of a listing of the $X_i$ and nothing else is printed.

Any number of decay records may be analyzed in sequence, and the computer will halt on a read instruction after the last decay record has been processed.

COMPUTER PROGRAM

```
        DIMENSION S(11),T(11),X(30)                                              001
      2 FORMAT(I4,I4,I4)                                                         002
      3 FORMAT(7F10.4)                                                           003
     10 READ2,N,IDENT,IX                                                         004
     11 DO12 I=1,N,7                                                             005
     12 READ3,X(I),X(I+1),X(I+2),X(I+3),X(I+4),X(I+5),X(I+6)                     006
        TEST=.0001                                                              007
        SET=0.                                                                  008
        K=0                                                                     009
     20 EA=(X(4)-X(2))/(X(1)-X(3))                                              010
        EB=(X(5)-X(3))/(X(2)-X(4))                                              011
        D=.5*(EA+EB)                                                            012
     35 DO36 I=1,11                                                             013
     36 S(I)=0.                                                                 014
     40 A=0.                                                                    015
     44 B=1.0                                                                   016
     45 DO49 I=1,N                                                              017
        S(1)=S(1)+X(I)*A                                                        018
        S(2)=S(2)+X(I)*B                                                        019
        S(3)=S(3)+X(I)                                                          020
        S(4)=S(4)+A*A                                                           021
        S(5)=S(5)+B*B                                                           022
        S(6)=S(6)+A*B                                                           023
        S(7)=S(7)+A                                                            024
        S(8)=S(8)+B                                                            025
        A=B-A                                                                   026
     49 B=-B*D                                                                  027
     50 S(9)=N                                                                  028
     55 DET=S(4)*(S(5)*S(9)-S(8)*S(8))-S(6)*(S(6)*S(9)-S(7)*S(8))               029
        DET=DET+S(7)*(S(6)*S(9)-S(8)*S(8))-S(5)*S(7))                           030
        DU=S(1)*(S(5)*S(9)-S(8)*S(8))-S(2)*(S(6)*S(9)-S(7)*S(8))                031
        DU=DU+S(3)*(S(6)*S(8)-S(5)*S(7))                                        032
```

21

```
      DV=S(4)*(S(2)*S(9)-S(3)*S(8))-S(1)*(S(6)*S(9)-S(7)*S(8))        033
      DV=DV+S(7)*(S(6)*S(3)-S(7)*S(2))                                034
   59 DW=S(4)*(S(5)*S(3)-S(2)*S(8))-S(6)*(S(6)*S(3)-S(7)*S(2))         035
      DW=DW+S(1)*(S(6)*S(8)-S(7)*S(5))                                 036
   60 U=DU/DET                                                        037
      V=DV/DET                                                        038
      W=DW/DET                                                        039
   64 Z=U/(1.+D)                                                      040
   65 DO68 I=1,11                                                     041
      T(1)=0.                                                         042
   68 S(I)=0.                                                         043
   69 CONTINUE                                                        044
   70 IF(SET)140,75,140                                               045
   75 T(1)=V                                                          046
      T(4)=1.                                                         047
      T(8)=U                                                          048
      T(9)=Z                                                          049
      T(10)=1.+D                                                      050
   79 T(11)=1.                                                        051
   80 DO 83 I=1,N                                                     052
      Q=T(1)+W-X(I)                                                   053
      S(1)=S(1)+Q*T(5)+T(2)*T(2)                                      054
      S(2)=S(2)+Q*T(6)+T(3)*T(2)                                      055
      S(3)=S(3)+Q*T(7)+T(4)*T(2)                                      056
      S(4)=S(4)+T(2)                                                  057
      S(5)=S(5)+T(3)*T(3)                                             058
      S(6)=S(6)+T(4)*T(3)                                             059
      S(7)=S(7)+T(3)                                                  060
      S(8)=S(8)+T(4)*T(4)                                             061
      S(9)=S(9)+T(4)                                                  062
      S(11)=S(11)-Q*T(2)                                              063
      T(5)=-2.*T(2)-D*T(5)                                            064
      T(6)=-T(3)-D*T(6)+T(11)                                         065
```

22

```
      T(7)=-T(4)-D*T(7)                                                              066
      T(2)=-T(9)-T(1)-D*T(2)                                                         067
      T(3)=-T(10)-D*T(3)                                                             068
      T(4)=-D*T(4)                                                                   069
      T(1)=-T(8)-D*T(1)                                                              070
      T(8)=-T(8)                                                                     071
      T(9)=-T(9)                                                                     072
      T(10) =-T(10)                                                                  073
   83 T(11)=-T(11)                                                                   074
   84 S(10)=N                                                                        075
   85 DA=S(5)*(S(8)*S(10)-S(9)*S(9))-S(6)*(S(6)*S(10)-S(7)*S(9))                      076
      DA=DA+S(7)*(S(6)*S(9)-S(7)*S(8))                                               077
      DB= S(2)*(S(8)*S(10)-S(9)*S(9))-S(6)*(S(3)*S(10)-S(4)*S(9))                     078
      DB=DB+S(7)*(S(3)*S(9)-S(4)*S(8))                                               079
      DC=S(2)*(S(6)*S(10)-S(7)*S(9))-S(5)*(S(3)*S(10)-S(4)*S(9))                      080
      DC=DC+S(7)*(S(3)*S(7)-S(4)*S(6))                                               081
      DD=S(2)*(S(6)*S(9)-S(7)*S(8))-S(5)*(S(3)*S(9)-S(4)*S(8))                        082
      DD=DD+6(6)*(S(3)*S(7)-S(4)*S(6))                                               083
   89 DELD =(S(11)*DA)/(S(1)*DA-S(2)*DB+S(3)*DC-S(4)*DD)                             084
   90 IF(100-K)156,156,91                                                            085
   91 K=K+1                                                                          086
      GO TO 95                                                                       087
   95 G=DELD/D                                                                       088
      IF(G)96,115,98                                                                 089
   96 G=-G                                                                           090
   98 IF(G-TEST)115,115,120                                                          091
  115 SET=1.0                                                                        092
      GO TO 120                                                                      093
  120 D=D+DELD                                                                       094
      GO TO 35                                                                       095
  140 SET=0.                                                                         096
  145 R=LOG(1./D)                                                                    097
      PSPRS=3.14159**2+R**2                                                          098
      DR=SQRT(R**2/PSPRS)                                                            099
  149 DEC=2.*R                                                                       100
```

```
150 TYPE 160,DR,DEC,Z,N,IDENT          101
151 TYPE161,D,V,W                       102
152 IF(IX) 153,10,153                   103
153 X(1)=V+W                            104
154 DO155I=2,N                          105
    X(I)=U-D*X(I-1)+(1.+D)*W            106
    U=-U                                107
155 CONTINUE                            108
156 DO157 I=1,N                         109
    TYPE161,X(I)                        110
157 CONTINUE                            111
158 GO TO 10                            112
160 FORMAT(2X,E14.5,2X,E14.5,2X,E14.5,2X,I8,2X,I8)   113
161 FORMAT(2X,E14.5,2X,E14.5,2X,E14.5,2X,E14.5)      114
    STOP                                115
    END                                 116
```

INPUT DATA

Card Columns

| | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 78 | 80 | Ident. | Card No. | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| N 12 | | | | | | | | | | 1 1 | 1 | (a) |
| Ident. 1 1 | 5.60 | -4.84 | 4.17 | -3.56 | 3.01 | -2.42 | 2.08 | | | 1 2 | 2 | (b) |
| | -1.68 | 1.32 | -1.00 | .71 | -.46 | | | | | 1 3 | 3 | (c) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| N 6 | 4 | 8 | 12 | | | | | |
| | 2 1 | | | | | | | 2 1 |
| 5.60 | -4.84 | 4.17 | -3.56 | 3.01 | -2.42 | | | 2 2 |

(a)  Control Card
(b)  Data Card containing $\bar{X}_0, X_1, \ldots, X_8$
(c)  Data Card containing $X_7, X_8, \ldots, X_{13}$

25

OUTPUT

| | N | Ident. |
|---|---|---|
| | 12 | 1 |

| $\Delta$ | $\ell$ | $\upsilon$ |
|---|---|---|
| 939.39219E-04 | 221.39778E-03 | 352.14725E-04 |
| 807.83017E-05 | 560.51995E-02 | 895.20820E-03 |
| h | $\overline{Y}_0$ | $\delta$ |

$\overline{Y}_i$

561.32778E-02
-483.17080E-02
416.26603E-02
-353.31033E-02
300.01387E-02
-249.24045E-02
206.84966E-02
-165.83908E-02
132.18807E-02
-990.01410E-03
723.54442E-03
-454.37846E-03

| | N | Ident. |
|---|---|---|
| | 6 | 2 |

| 162.94379E-03 | 153.89301E-03 | 244.85510E-04 |
|---|---|---|
| 124.69141E-04 | 558.14592E-02 | 925.93932E-03 |

559.39283E-02
-484.18036E-02
419.34113E-02
-354.50098E-02
299.26590E-02
-243.31860E-02